International Journal of Theoretical Physics, Vol. 32, No. 10, 1993

Measure on the Inductive Limit of Projection Lattices

M. S. Matvejchuk[1]

Received April 30, 1993

A probability measure on a nondecreasing net of lattices of orthogonal projections in von Neumann algebras is extended to a probability on the inductive limit of the lattices.

Let $\{\mathfrak{A}_\alpha\}$ be a nondecreasing net of von Neumann algebras acting in a Hilbert space $\mathscr{H}$ and $\mathfrak{A}$ be the von Neumann algebra generated by the family $\mathfrak{A}_\alpha$, i.e., $\mathfrak{A} = (\bigcup_\alpha \mathfrak{A}_\alpha)''$. The algebra $\mathfrak{A}$ is called the inductive limit of the set $\{\mathfrak{A}_\alpha\}$. By analogy, the lattice $\mathfrak{A}^\Pi$ of all orthogonal projections from $\mathfrak{A}$ is called the inductive limit of the lattices $\{\mathfrak{A}_\alpha^\Pi\}$ of all orthogonal projections from $\{\mathfrak{A}_\alpha\}$. We call a function μ: $\bigcup_\alpha \mathfrak{A}_\alpha^\Pi \to \mathbb{R}^+$ a probability measure provided $\mu(\mathrm{I}) = 1$ and $\mu(p) = \sum_\beta \mu(p_\beta)$, wherein $p = \sum_\beta p_\beta$ and $p, p_\beta \in \bigcup_\alpha \mathfrak{A}_\alpha^\Pi$. The latter condition is essential even in the classical case.

Note that in the proof of the following theorem we will not use Gleason's theorem or its analogs.

Theorem. Let a von Neumann algebra $\mathfrak{A}$ of countable type not containing any type I_2 direct summand be the inductive limit of von Neumann algebras $\{\mathfrak{A}_\alpha\}$. Then any probability measure μ: $\bigcup_\alpha \mathfrak{A}_\alpha^\Pi \to [0, 1]$ can be extended to a probability on $\mathfrak{A}^\Pi$.

The proof will consist of several steps.

(i) Let us establish the existence of reduced subalgebras to which we will extend μ by a strong continuity. Let $\mathscr{S}(a) \equiv \{q \in \bigcup_\alpha \mathfrak{A}_\alpha^\Pi : \mu(q) > a\varphi(q)\}$. Here and in what follows φ is a faithful normal state on $\mathfrak{A}$. Take an arbitrary projection $(0 \neq)\, p \in \bigcup_\alpha \mathfrak{A}_\alpha^\Pi$ and choose a number $t > 0$ such that $t\varphi(p) > \mu(p)$. Then for every projection $q \in \mathscr{S}(t)$ with $q \leq p$ we have

[1]Department of Mathematics and Mechanics, Kazan State University, 420008 Kazan, Russia.

$t\varphi(q) < \mu(q)$. We fix an arbitrary $\varepsilon \in (0, 1)$. Choose a projection p_1 $(\equiv p_1^t) \in \mathscr{S}(t)$, $p_1 \leq p$, with

$$\sup\{\mu(q); q \in \mathscr{S}(t), q \leq p\} - \mu(p_1) < \varepsilon$$

By the definition of p_1 the conditions $q \in \mathscr{S}(t)$, $q \perp p_1$, and $q \leq p$ imply $0 \leq t\varphi(q) < \mu(q) < \varepsilon$. Next, choose a projection p_2 $(\equiv p_2^t) \in \mathscr{S}(t)$, $p_2 \perp p_1$ such that $p_2 \leq p$ and

$$\sup\{\mu(q); q \in \mathscr{S}(t), q \leq p - p_1\} - \mu(p_2) < \varepsilon^2$$

Let us continue the process by induction. If $p_1, \ldots, p_n$ are already defined, then take a projection p_{n+1} $(\equiv p_{n+1}^t)$ in $\mathscr{S}(t)$ (if one exists) with $p_{n+1} \leq p - \sum_{i=1}^n p_i$ and

$$\sup\left\{\mu(q); q \in \mathscr{S}(t), q \leq p - \sum_{i=1}^n p_i\right\} - \mu(p_{n+1}) < \varepsilon^{n+1}$$

By the construction, we have

$$0 \leq t\varphi(p_{n+1}) < \mu(p_{n+1}) < \varepsilon^n \tag{1}$$

If the process stops at step n, then we put $p_m = 0$ for all $m > n$.

We denote by $G_t(p)$ the projection $p - \sum_{m=1}^\infty p_m$. By the definition of a number t we have $G_t(p) \neq 0$. In general, $G_t(p) \notin \bigcup_\alpha \mathfrak{A}_\alpha^\Pi$. The sequence of the projections $\{e_m(t)\}_{m=1}^\infty$ where $e_m(t) \equiv p - \sum_{n=1}^m p_n$ is called determining for $G_t(p)$. By the inequality (1),

$$\varphi(p) \geq \varphi(G_t(p)) = \varphi(p) - \sum_{m=1}^\infty \varphi(p_m) > \varphi(p) - \frac{1}{t(1-\varepsilon)} \tag{2}$$

By the definition of projections $e_m(t)$, for every projection $e_1 \leq e_m(t)$, $e_i \in \bigcup_\alpha \mathfrak{A}_\alpha^\Pi$ we obtain $\mu(e_i) \leq t\varphi(e_i) + \varepsilon^m$. Let $\{e_i\}_{i=1}^k \subset \bigcup_\alpha \mathfrak{A}_\alpha^\Pi$ be an orthogonal family, $e \equiv \sum_{i=1}^k e_i \leq e_m(t)$ and $\varepsilon_i \equiv \mu(e_i) - t\varphi(e_i)$. Then

$$\sum_i \varepsilon_i = \sum_i (\mu(e_i) - t\varphi(e_i)) = \mu(e) - t\varphi(e) < \varepsilon^m$$

If all numbers $\lambda_i > 0$, then

$$\sum_i \lambda_i \mu(e_i) = t \sum_i \lambda_i \varphi(e_i) + \sum_i \lambda_i \varepsilon_i \leq t \sum_i \lambda_i \varphi(e_i) + \varepsilon^m \max_i \lambda_i$$

Thus the following remark is valid.

Remark. Let $a = \int \lambda \, de_\lambda^a$ be the spectral decomposition of the operator $a \in \bigcup_\alpha \mathfrak{A}_\alpha^\Pi$. If $a \leq \lambda e_m(t)$ for some $\lambda > 0$, then $\hat{\mu}(a) \equiv \int \lambda \, d\mu(e_\lambda^a) \leq t\varphi(a) + \|a\| \varepsilon^m$.

Lemma 1. For every projection $\bar{p} \in \bigcup_\alpha \mathfrak{A}_\alpha^\Pi$ ($\bar{p} \neq 0$) there exists a family $G_{t_n}(p_n)$ ($p_n \leq \bar{p}$) of mutually orthogonal projections such that $\sum_n G_{t_n}(p_n) = \bar{p}$ and for every n_0 of its initial members there exist numbers $N(t_n, n_0)$ such that for $m > N(t_n, n_0)$ the projections $e_m(t_k)$ in the determining sequence for $G_{t_k}(p_k)$ are mutually orthogonal for distinct $k = 1, 2, \ldots, n_0$.

Proof. Take any projection $G_t(\bar{p})$ ($p_1 \equiv \bar{p}$, $t_1 \equiv t$) as the first member. If $G_t(\bar{p}) < \bar{p}$, then choose a number m such that for the projection $p_2 \equiv \sum_{n=1}^m p_n^{t_1}$ $[= \bar{p} - e_m(t_1)]$ the inequality $2/3\varphi[\bar{p} - G_{t_1}(p_1)] < \varphi(p_2)$ is fulfilled. By virtue of inequality (2) a number t_2 can be found such that

$$\varphi(G_{t_2}(p_2)) \geq 1/2\varphi[\bar{p} - G_{t_1}(p_1)]$$

Obviously $G_{t_1}(p_1) \perp G_{t_2}(p_2)$. Let us continue the process by induction. Let the projections $G_{t_1}(p_1), \ldots, G_{t_n}(p_n)$ be already defined and $\varphi(\bar{p} - \sum_{i=1}^n G_{t_i}(p_i)) > 0$. Also, let $\{e_m(t_i)\}_{m=1}^\infty$ be the determining sequence for $G_{t_i}(p_i)$ ($1 \leq i \leq n$). Choose numbers $m_1, \ldots, m_n$ such that

$$\varphi\left(\bar{p} - \sum_{j=1}^n p_{m_j}(t_j)\right) > 2/3\varphi\left(\bar{p} - \sum_{j=1}^n G_{t_j}(p_j)\right)$$

and denote by p_{n+1} the projection $\bar{p} - \sum_{j=1}^n e_{m_j}(t_j)$. Next, find a number t_{n+1} such that $\varphi(G_{t_{n+1}}(p_{n+1})) > 1/2\varphi(\bar{p} - \sum_{i=1}^n G_{t_i}(p_i))$. Thus

$$\varphi\left(\bar{p} - \sum_{i=1}^{n+1} G_{t_i}(p_i)\right) \leq \frac{1}{2}\varphi\left(\bar{p} - \sum_{i=1}^n G_{t_i}(p_i)\right) \leq \frac{1}{2^n}\varphi(\bar{p} - G_{t_1}(p_1)) \xrightarrow[n\to\infty]{} 0$$

By the definition, the sequence $\{G_{t_n}(p_n)\}$ satisfies the assertion of the lemma. The process will stop at the step k only if $\bar{p} = \sum_{i=1}^k G_{t_i}(p_i)$ and $e_{m_i}(t_i) = G_{t_i}(p_i)$ for some m_i for all $1 \leq i \leq k$. Thus the family $G_{t_i}(p_i)$ is suitable. The lemma follows.

We denote by $\mathfrak{N}$ the set of projections $q \in \mathfrak{A}^\Pi$ satisfying $q \leq \sum_{m=1}^n G_{t_m}(p_m)$ for some orthogonal family $\{G_{t_m}(p_m)\}$ as in the lemma 1.

(ii) Extending μ to the projections in $\mathfrak{N}$. Let the projections $e, f \in \mathfrak{A}^\Pi$ and $\Delta \equiv \varphi(ef^\perp e)$. By the construction used in Gunson (1972), there exist decompositions $e = e_0 + e_1$, $f = f_0 + f_1$, $e_0, e_1, f_0, f_1 \in \mathfrak{A}^\Pi$ such that $\varphi(e_1) \leq \Delta$, $\varphi(f_1) \leq \varphi^{1/2}((e-f)^2) + \Delta + \Delta^{1/2}$, and $\|e_0 - f_0\| \leq \Delta^{1/2}$.

Let projection $G_t(p)$ be arbitrary and let $\{e_m(t)\}$ be the determining sequence for $G_t(p)$. By an analog of the Theorem 2.11 of Gunson (1972), for all projections $e, f \in \bigcup_\alpha \mathfrak{A}_\alpha^\Pi$ with $e, f \leq e_m(t)$ we have

$$\begin{aligned} |\mu(e) - \mu(f)| &\leq |\mu(e_0) - \mu(f_0)| + \mu(e_1) + \mu(f_1) \\ &\leq 3.8^{1/2}\|e_0 - f_0\| + \mu(e_1) - \mu(f_1) \\ &\leq 3.8^{1/2}\Delta^{1/2} + t\varphi(e_1) + \varepsilon^m + t\varphi(f_1) + \varepsilon^m \end{aligned} \quad (3)$$

Let now $\{G_{t_n}(p_n)\}$ be an orthogonal family as in Lemma 1 and let $\{e_m(t_n)\}_{m=1}^{\infty}$ be the determining sequence for $G_{t_n}(p_n)$. Inequalities (3) show that for a projection $e \in \mathfrak{A}^{\Pi}$, $e \leq \sum_{i=1}^{n_0} G_{t_i}(p_i)$ and for every sequence of projections $\{g_m\} \subset \bigcup_\alpha \mathfrak{A}_\alpha^{\Pi}$, $g_m \leq \sum_{i=1}^{n_0} e_m(t_i)$ that is strongly convergent to $e\,(g_m \to^s e)$ there exists $\lim_{m\to\infty} \mu(g_m) \equiv \tilde{\mu}(e)$ which is independent of $\{g_m\}$ (the sequence $\{g_m\}$ is called determining for e). Moreover, $\tilde{\mu}(e) \leq 2\sum_{i=1}^{n_0} t_i \varphi(e_m(t_i) e e_m(t_i))$. This implies $\tilde{\mu}$ to be strongly continuous in $\mathcal{O}$ on the projections of the reduced algebra $\mathfrak{A}_G$, where $G \equiv \sum_{i=1}^{n_0} G_{t_i}(p_i)$. If $e_1, e_2 \in \mathfrak{A}^{\Pi}$, $e_1 \perp e_2$, and $g_m \to^s e = e_1 + e_2$, then there exist sequences $\{g'_m\}$ and $\{g''_m\}$ from $\bigcup_\alpha \mathfrak{A}_\alpha^{\Pi}$ such that $g_m = g'_m + g''_m$ and $g'_m \to^s e_1$. Hence $\tilde{\mu}(e_1 + e_2) = \tilde{\mu}(e_1) + \tilde{\mu}(e_2)$. Thus $\tilde{\mu}$ is a countably additive measure on $\mathfrak{A}_G^{\Pi}$. Obviously $\tilde{\mu}(e)$ does not depend upon projection $\bar{p}$ and a family of the projections $\{G_{t_n}(p_n)\}$ satisfying $e \leq \sum_{i=1}^{k} G_{t_i}(p_i)$, $\bar{p} = \sum_i G_{t_i}(p_i)$.

(iii) Extending μ to the lattice $\mathfrak{A}^{\Pi}$. Let $\{G_{t_n}(p_n)\}$ be a family of projections as in Lemma 1 for $\bar{p} = I$. If the family $\{G_{t_n}(p_n)\}$ is finite, then the function $\tilde{\mu}$ obviously is a suitable extension of the measure μ. Now suppose that the projection $G_m \equiv \sum_{k=1}^{m} G_{t_k}(p_k) \neq I$ for all m. Then the set of the projections $\mathfrak{M} \equiv \bigcup_{m=1}^{\infty} \mathfrak{A}_{G_m}^{\Pi}$ is an ideal of the projections, i.e:

1. $p \in \mathfrak{M}$, $g \in \mathfrak{A}^{\Pi}$, $g \leq p \Rightarrow g \in \mathfrak{M}$.
2. $p, g \in \mathfrak{M}$, $\|pg\| < 1 \Rightarrow p \vee g \in \mathfrak{M}$.
3. $\sup_{p \in \mathfrak{M}} p = 1$.

The function $\tilde{\mu}$ is a measure on it. By a theorem of Matvejchuk (1983) $\tilde{\mu}$ can be uniquely extended to a measure $\bar{\mu}$ on $\mathfrak{A}^{\Pi}$. Let us show $\bar{\mu}$ to be a suitable extension. We first establish an analog to the inequality $|\mu(e) - \mu(f)| \leq 3.8^{1/2}\|e - f\|$, $\forall e, g \in \sum_\alpha \mathfrak{A}_\alpha^{\Pi}$, for the function $\tilde{\mu}$. Let the projections $e, f \in \mathfrak{N}$ and $\{e_m\}$, $(e_m \to^s e)$, $\{f_m\}$, $(f_m \to^s f)$ be the determining sequences. By the constructions, for every $\varepsilon > 0$ projections e_m and f_m can be chosen such that $\|e_m - f_m\| \leq \|e - f\| + \varepsilon$ for all m. Then

$$
\begin{aligned}
|\tilde{\mu}(e) - \tilde{\mu}(f)| &= \lim_{m\to\infty} |\mu(e_m) - \mu(f_m)| \\
&\leq 3.8^{1/2} \overline{\lim_{m\to\infty}} \|e_m - f_m\| \\
&\leq 3.8^{1/2}(\|e - f\| + \varepsilon) \qquad (4)
\end{aligned}
$$

Let now a sequence $\{e_m\} \subset \mathfrak{A}^{\Pi}$ be such that $e_m \to^s e \in \mathfrak{N}$, where $e_m \leq G_m$ for all m. Let $e = e_0 + e_1$ and $e_m = e_m^0 + e_m^1$ be expansions with $e_1 \to^s_{m\to\infty} 0$, $e_m^1 \to^s_{m\to\infty} 0$, and $\|e_0 - e_m^0\| \to_{m\to\infty} 0$. Since $\tilde{\mu}$ is a countably additive measure on $\mathfrak{A}_e^{\Pi}$, it follows that $\tilde{\mu}(e_1) \to_{m\to\infty} 0$. Similarly, $\bar{\mu}(e_m^1) = \tilde{\mu}(e_m^1) \to_{m\to\infty} 0$. By virtue of inequality (4), $\tilde{\mu}(e_m^0) - \tilde{\mu}(e_0) \to_{m\to\infty} 0$. Thus $\tilde{\mu}(e) = \lim \tilde{\mu}(e_0) = \lim \tilde{\mu}(e_m^0) = \lim \tilde{\mu}(e_m) = \lim \bar{\mu}(e_m) = \bar{\mu}(e)$. Therefore, $\bar{\mu}$ is an extension of $\tilde{\mu}$ from $\mathfrak{N}$. Let $p \in \bigcup_\alpha \mathfrak{A}_\alpha^{\Pi}$.

Obviously $\mu(p) \geq \sup\{\tilde{\mu}(q): q \in \mathfrak{N}, q \leq p\}$. The process we used in the proof of Lemma 1 enables us to obtain a sequence $\{q_n\} \subset \bigcup_\alpha \mathfrak{A}_\alpha^\Pi$ of mutually orthogonal projections satisfying $\sum_n q_n = p$ and $|\mu(q_n) - \tilde{\mu}(G_{t_n}(q_n))| < \varepsilon^n$ for some $t_n > 0$. Since $\sum_n G_{t_n}(q_n) \leq \sum_n q_n = p$, it follows that

$$\mu(p) \geq \sum_n \tilde{\mu}(G_{t_n}(q_n)) \geq \sum_n (\mu(q_n) - \varepsilon^n) = \sum_n \mu(q_n) - \frac{\varepsilon}{1-\varepsilon} = \mu(p) = \frac{\varepsilon}{1-\varepsilon}$$

Since ε is arbitrary, we have

$$\mu(p) = \sup\{\tilde{\mu}(q): q \in \mathfrak{N}, q \leq p\} = \sup\{\bar{\mu}(q): q \in \mathfrak{N}, q \leq p\} = \bar{\mu}(p)$$

REFERENCES

Gunson, J. (1972). Physical states on quantum logics, I, *Annals de l'Institut Henri Poincaré A*, **17**(4), 295–230.

Matvejchuk, M. S. (1983). Probability measure on an ideal of projections, *Veroiatnostnye metody i kibernetika* (*Kazan*), **19**, 51–55 [in Russian].